Complex Analysis

Sec 50. Cauchy Integral Formula

Theorem Let f be analytic everywhere inside and on a simple closed contour C, taken in the positive sense. If z_0 is any point interior to C, then

$$f(z_0) = \frac{1}{2\pi i} \int_C \frac{f(z)}{z - z_0}\, dz \qquad \text{—— } (*)$$

Before doing the proof of the above theorem, let us understand it through some examples.

Remark:- From (*), we observe that

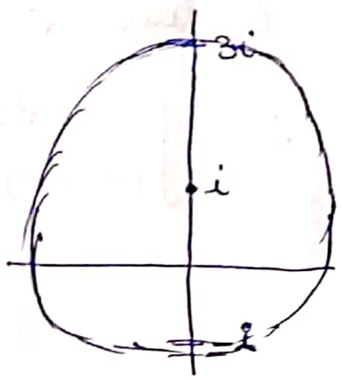
$$\int_C \frac{f(z)}{z - z_0}\, dz = f(z_0) \cdot 2\pi i$$

So, Cauchy Integral formula can be used to evaluate certain integrals along simple closed contours.

Example 1. Evaluate $\int_C \frac{1}{z^2+4}\, dz$ where $C : |z - i| = 2$.

Now, $\displaystyle\int \frac{1}{z^2+4}\, dz = \int \frac{dz}{(z-2i)(z+2i)}$

$$= \int \frac{\frac{1}{z+2i}}{(z-2i)}\, dz$$



(Note that $z = 2i$ lies inside C)

$$= \int \frac{f(z)}{(z-2i)}\,dz \quad \text{where } f(z) = \frac{1}{z+2i}$$

and $z_0 = 2i$

$$= 2\pi i \times f(z_0) \quad \text{(By Cauchy Integral Formula)}$$

$$= 2\pi i \cdot f(2i)$$

$$= 2\pi i \cdot \frac{1}{(2i+2i)}$$

$$= 2\pi i \times \frac{1}{4i}$$

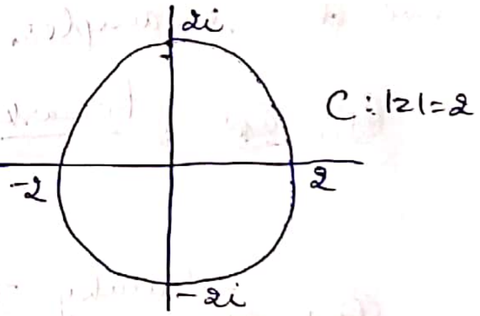
$$= \pi/2$$

Example 2. Evaluate $\displaystyle\int_C \frac{z\,dz}{(9-z^2)(z+i)}$ where $C: |z|=2$

Let us factorize the denominator

$$(9-z^2)(z+i) = -9(z^2-9)(z+i)$$

$$= -9(z-3)(z+3)(z+i)$$



We note that only $z=-i$ is interior to C.

So Let $f(z) = \dfrac{z}{9-z^2}$

Then f is analytic inside and on the contour $C: |z|=2$.

$$\therefore \int \frac{z\,dz}{(9-z^2)(z+i)} = \int_C \frac{\left(\frac{z}{9-z^2}\right)}{(z+i)}\,dz$$

$$= 2\pi i \times f(-i) = 2\pi i \cdot \frac{-i}{9-(-i)^2}$$

$$= \pi/5$$

Scanned with CamScanner

~~Background needed to~~

We shall use the following ~~ideas~~ background to prove Cauchy Integral Formula.

1. <u>Cauchy-Goursat Theorem</u>

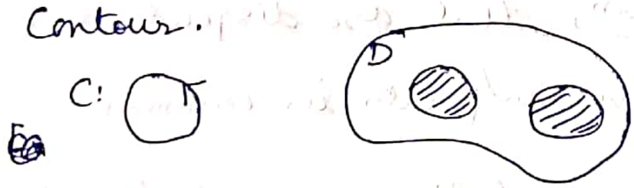
If a function f is analytic at all points interior to and on a simple closed contour C, then

$$\int_C f(z)\,dz = 0$$

2. ~~Defn~~ <u>Simply Connected Domain</u>

A <u>simply connected domain</u> D is a domain such that <u>every simple closed contour within it encloses only points of D.</u>

<u>Example 1.</u> The set of points inside to a simple closed contour.



<u>Example 2.</u> The ~~annular~~ domain D is ~~between two concentric circles~~ <u>not simply connected</u>

3. <u>Theorem:</u> If a function f is analytic throughout a simply connected domain D, then

$$\int_C f(z)\,dz = 0$$

for every closed contour C lying in D.

4. <u>Multiply connected domains</u> (MCD)

A domain that is not simply connected is s.t.b. multiply connected domain.

5. <u>Adaption of Cauchy-Goursat Theorem</u> (MCD)

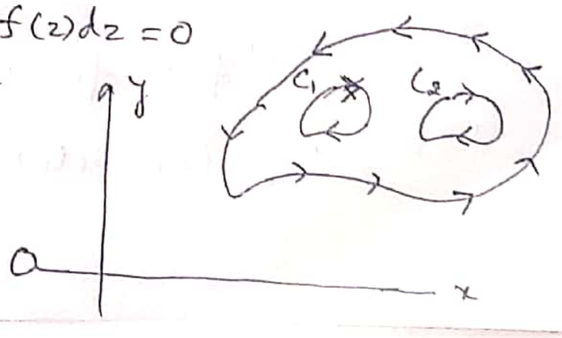
<u>Theorem:</u> Suppose that

(a) C is a simple closed contour described in the <u>Counter clockwise direction</u>;

(b) C_k $(k=1,2,\ldots,n)$ are simple closed contours ~~inside~~ interior to C, all described in the <u>clockwise direction</u>, that are disjoint and whose interiors have no points in common

If a function f is analytic on all ~~the~~ of these contours and throughout the multiply connected domain consisting of the points <u>inside C</u> and <u>exterior to each C_k</u>, then

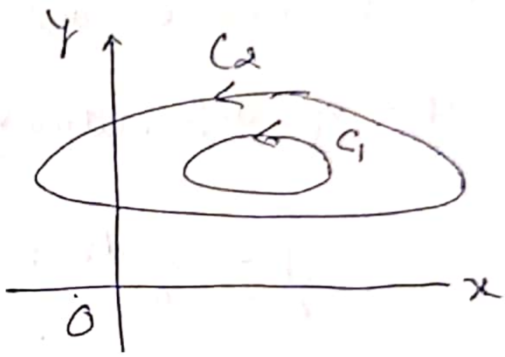
$$\int_C f(z)\,dz + \sum_{k=1}^{n} \int_{C_k} f(z)\,dz = 0$$



6. Corollary to 5. (Principle of Deformation of Paths).

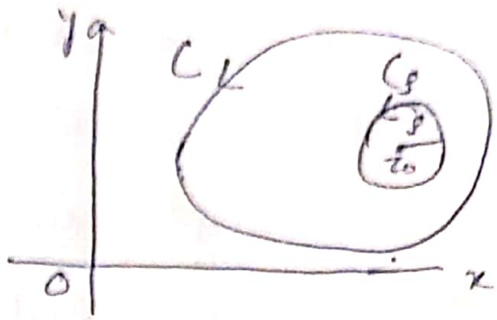
Let C_1 & C_2 denote positively oriented simple closed contours, where C_1 is interior to C_2. If a function f is analytic in the closed region consisting of those contours and all points between them, then

$$\int_{C_2} f(z)\,dz = \int_{C_1} f(z)\,dz.$$



Proof of Cauchy Integral Formula

Let C_ρ be ~~denote~~ a +vely oriented circle $|z-z_0|=\rho$ where 'ρ' is small enough that C_ρ is interior to C.



Since the quotient $\frac{f(z)}{(z-z_0)}$ is analytic between and on the contours C and C_ρ, therefore from the principle of deformation of paths, we have

$$\int_C \frac{f(z)}{(z-z_0)}dz = \int_{C_\rho} \frac{f(z)}{(z-z_0)}dz \quad -(1)$$

Subtract $f(z_0)\int_{C_\rho}\frac{dz}{(z-z_0)}$ on both sides of (1)

$$\int_C \frac{f(z)}{(z-z_0)}dz - f(z_0)\int_{C_\rho}\frac{dz}{(z-z_0)} = \int_{C_\rho}\frac{f(z)-f(z_0)}{(z-z_0)}dz$$

$$\int_C \frac{f(z)}{(z-z_0)}dz - f(z_0).2\pi i = \int_{C_\rho}\frac{f(z)-f(z_0)}{(z-z_0)}dz \quad -(2)$$

$\left(\because \int_{C_\rho}\frac{dz}{(z-z_0)} = 2\pi i\right)$

Verify it

Now, since f is analytic inside and on C,

$\therefore$ f is analytic at z_0

$\therefore$ f is continuous at z_0.

So, given $\varepsilon > 0$ $\exists$ $\delta > 0$ such that

$$|f(z) - f(z_0)| < \varepsilon \quad \text{whenever } |z - z_0| < \delta$$

Assume that $\rho < \delta$, where ρ is radius of C_ρ

Then, For $z \in C_\rho$

$$|f(z) - f(z_0)| < \varepsilon \quad \text{whenever } |z - z_0| = \rho < \delta$$

$\therefore$ By M-L Theorem

$$\left| \int_{C_\rho} \frac{f(z) - f(z_0)}{z - z_0} dz \right| < \frac{\varepsilon \times 2\pi\rho}{\rho} = 2\pi\varepsilon \quad \text{—(3)}$$

$\therefore$ From (2) and (3)

$$\left| \int_C \frac{f(z)}{z - z_0} dz - 2\pi i f(z_0) \right| < 2\pi\varepsilon$$

Since $\varepsilon > 0$ is arbitrary,

$$\therefore \left| \int_C \frac{f(z)}{z - z_0} dz - 2\pi i f(z_0) \right| = 0$$

$$\Rightarrow \int_C \frac{f(z)}{z - z_0} dz - 2\pi i f(z_0) = 0$$

$$\Rightarrow \boxed{\int_C \frac{f(z)}{z - z_0} dz = 2\pi i f(z_0)}$$

or $$\boxed{f(z_0) = \frac{1}{2\pi i} \int_C \frac{f(z)}{z - z_0} dz}$$

Sec 51.

An Extension of the Cauchy Integral Formula

$$\int_C \frac{f(z)}{(z-z_0)^{n+1}}\,dz = \frac{2\pi i}{n!} f^{n}(z_0)$$

$(n=0,1,2,3,\ldots)$

Where f is analytic inside and on a simple closed contour C, taken in the +ve sense.

Proof. We are given that f is analytic everywhere inside and on a simple closed contour C.

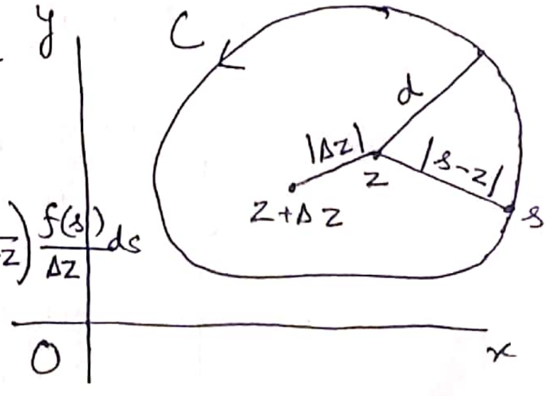
Then by Cauchy Integral Formula,

$$f(z) = \frac{1}{2\pi i}\int_C \frac{f(s)\,ds}{(s-z)} \quad \text{—(1)}$$

where 'z' is interior to C and where 's' denote points on C.

We Claim that $f'(z) = \frac{1}{2\pi i}\int_C \frac{f(s)}{(s-z)^2}\,ds$

Let 'd' denote the smallest distance from 'z' to points 's' on C



By (1),

$$\frac{f(z+\Delta z)-f(z)}{\Delta z} = \frac{1}{2\pi i}\int_C \left(\frac{1}{s-(z+\Delta z)} - \frac{1}{s-z}\right)\frac{f(s)}{\Delta z}\,ds$$

$$= \frac{1}{2\pi i}\int_C \frac{f(s)\,ds}{(s-z-\Delta z)(s-z)} \quad \text{—(2)}$$

where $0 < |\Delta z| < d$.

Then,

$$\frac{f(z+\Delta z)-f(z)}{\Delta z} - \frac{1}{2\pi i}\int_C \frac{f(s)\,ds}{(s-z)^2}$$

$$= \frac{1}{2\pi i}\int_C \frac{f(s)\,ds}{(s-z-\Delta z)(s-z)} - \frac{1}{2\pi i}\int_C \frac{f(s)\,ds}{(s-z)^2} \quad (\text{By } 2)$$

$$= \frac{1}{2\pi i}\int_C \frac{\Delta z\, f(s)\,ds}{(s-z-\Delta z)(s-z)^2} \quad -3$$

Let M denote the maximum value of $|f(s)|$ on C

(M exists, because f is analytic
$\Rightarrow f$ is cts on C
$\Rightarrow f$ is bdd on C)

Also, we observe that

$$|s-z| \geq d \quad -4 \qquad \text{and} \qquad |\Delta z| < d \quad -5$$

$$\therefore\ |s-z-\Delta z| = |(s-z)-\Delta z|$$
$$\geq \big|\,|(s-z)| - |\Delta z|\,\big|$$
$$\geq d - |\Delta z| \quad (\text{from } 4)$$
$$> 0 \quad (\text{from } 5) \quad -6$$

$\therefore$ By M-L Theorem

$$\left| \int_C \frac{\Delta z\, f(s)\,ds}{(s-z-\Delta z)(s-z)^2} \right| \leq \frac{|\Delta z|\, M}{(d-|\Delta z|)\, d^2} \quad (\text{From } 4, 5, 6)$$

Let $\Delta z \to 0$,

We get

$$\lim_{\Delta z \to 0} \frac{f(z+\Delta z) - f(z)}{\Delta z} - \frac{1}{2\pi i}\int_C \frac{f(s)\,ds}{(s-z)^2} = 0$$

$$\therefore \quad f'(z) = \frac{1}{2\pi i}\int_C \frac{f(s)\,ds}{(s-z)^2}$$

Similarly, using the above technique repeatedly,

$$f^n(z) = \frac{n!}{2\pi i}\int_C \frac{f(s)\,ds}{(s-z)^{n+1}}, \quad (n=1,2,3,\ldots) \qquad \text{--- (6)}$$

~~Consider~~

Considering $f^0(z) = f(z)$ and $0! = 1$

Expression (6) is also valid when $n=0$, in which case it becomes <u>Cauchy Integral Formula</u>

$$\therefore \quad \boxed{\int_C \frac{f(z)\,dz}{(z-z_0)^{n+1}} = \frac{2\pi i}{n!} f^n(z_0), \quad (n=0,1,2,3,\ldots)}$$

<u>Example</u> Let $C: |z|=1$, +vely oriented

$$f(z) = e^{2z}$$

$$\int_C \frac{e^{2z}}{z^4}\,dz = \int_C \frac{f(z)}{(z-0)^{3+1}}\,dz$$

$f(z) = e^{2z}$ is analytic everywhere (i.e. inside and on C)

$$\therefore \int \frac{e^{2z}}{z^4}\,dz = \int_C \frac{f(z)\,dz}{(z-0)^{3+1}} = \frac{2\pi i}{3!} f'''(0)$$

$$= \frac{2\pi i}{6} \times 8$$

$$= \frac{8\pi i}{3}.$$

Example 2. Let 'z_0' be any point interior to a +vely oriented simple closed contour C.

Evaluate (a) $\displaystyle\int_C \frac{dz}{(z-z_0)}$ (b) $\displaystyle\int_C \frac{dz}{(z-z_0)^{n+1}}$ $(n = 1, 2, 3, \ldots)$

(a) Here $f(z) = 1$ which is clearly analytic inside and on the contour C.

$$\therefore \int \frac{dz}{(z-z_0)} = \int \frac{f(z)\,dz}{(z-z_0)} = 2\pi i\, f(z_0)$$

$$= 2\pi i \cdot 1$$

$$= 2\pi i.$$

(b) $$\int \frac{dz}{(z-z_0)^{n+1}} = \int \frac{f(z)\,dz}{(z-z_0)^{n+1}} = \frac{2\pi i}{n!} f^{n}(z_0)$$

$$= \frac{2\pi i}{n!} \times 0 = 0$$

$$\left(\begin{array}{l} \because f(z) = 1 \ \forall z \\ \therefore f^{n}(z) = 0 \ \forall n = 1, 2, \ldots \end{array} \right)$$